JOURNAL OF APPROXIMATION THEORY 72, 193–209 (1993)

Relative Asymptotics for Orthogonal Polynomials with a Sobolev Inner Product*

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Communicated by A. P. Magnus

Received December 28, 1990; accepted in revised form December 3, 1991

We investigate orthogonal polynomials for a Sobolev type inner product $\langle f, g\rangle = (f, g) + \lambda f'(c)\, g'(c)$, where (f, g) is an ordinary inner product in $L_2(\mu)$ with μ a positive measure on the real line. We compare the Sobolev orthogonal polynomials with the orthogonal polynomials corresponding to the measure μ and analyse the five-term recurrence relation for the Sobolev orthogonal polynomials.

1. INTRODUCTION

Suppose μ_k $(k = 0, 1, 2, ..., m)$ are positive measures on the interval (a, b) and introduce the Sobolev inner product

$$\langle f, g\rangle = \sum_{k=0}^{m} \int_a^b f^{(k)}(x)\, g^{(k)}(x)\, d\mu_k(x), \tag{1}$$

where f, g are in the Sobolev space

$$W_{2,m}[(a, b); \mu_0, ..., \mu_m] = \{f \in A_m(a, b) \cap L_2[(a, b); \mu_0, ..., \mu_{m-1}] : f^{(m)} \in L_2[(a, b); \mu_m]\}.$$

* This research was partially supported by a research grant of the Belgian National Fund for Scientific Research and the convention Katholieke Universiteit Leuven/Consejo Superior de Investigaciones Científicas.

† Research Associate of the Belgian National Fund for Scientific Research.

640/72/2-5

0021-9045/93 $5.00

As usual $A_m(a, b)$ is the function space containing all functions $f: (a, b) \to \mathbb{C}$ such that $f \in C^{(m-2)}$ and $f^{(m-1)}$ is absolutely continuous on (a, b). We consider the orthonormal polynomials $q_n(x)$ $(n = 0, 1, 2, ...)$ which are associated to this inner product,

$$\langle q_n, q_m \rangle = \delta_{m,n}. \tag{2}$$

Sobolev type orthogonal polynomials already appear in work of Lewis [10] and were later considered by Althammer [1] and further investigated by Brenner [4], Schäfke [14], Iserles *et al.* [7, 8], and others. We are particularly interested in the case where $\mu_0 = \mu$ is a positive measure for which the orthogonal polynomials are known and μ_k $(k = 1, 2, ..., m)$ are measures with all their mass concentrated at one point $c \in \mathbb{R}$. This particular case has also been studied previously by Koekoek [9], Bavinck and Meijer [3, 2], and Marcellán and Ronveaux [11]. Our goal is to compare the Sobolev orthogonal polynomials with the ordinary orthogonal polynomials associated with the measure μ in order to investigate how the addition of the derivatives in the inner product influences the orthogonal system. We will emphasize the asymptotic behaviour of the Sobolev orthogonal polynomials relative to the ordinary orthogonal polynomials associated with the measure μ. In order to do this we will assume that μ is a measure for which the asymptotic behaviour of the orthogonal polynomials is known, and the most relevant class here is the Nevai class $M(0, 1)$ of orthogonal polynomials with converging recurrence coefficients.

2. Comparison of Orthogonal Polynomials

We consider the simple case where the Sobolev inner product is

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, d\mu(x) + \lambda f'(c)\, g'(c), \tag{3}$$

where $c \in \mathbb{R}$, $\lambda > 0$, and $\mu \in M(0, 1)$. Recall that the Nevai class $M(0, 1)$ consists of all measures μ for which the corresponding orthonormal polynomials $p_n(x)$ satisfy the three-term recurrence relation

$$xp_n(x) = a_{n+1}^0 p_{n+1}(x) + b_n^0 p_n(x) + a_n^0 p_{n-1}(x) \tag{4}$$

with coefficients satisfying

$$\lim_{n \to \infty} a_n^0 = \tfrac{1}{2}, \qquad \lim_{n \to \infty} b_n^0 = 0.$$

Denote by $q_n(x)$ $(n = 0, 1, 2, ...)$ the orthonormal polynomials for the

Sobolev inner product (3) and by $p_n(x)$ $(n=0, 1, 2, ...)$ the orthonormal polynomials corresponding to the measure μ. Our aim is to investigate the relationship between both systems of orthonormal polynomials. The situation is very similar to adding a mass point distribution to the measure μ and comparing the corresponding polynomials [12, pp. 131–133] and surprisingly similar results are valid.

THEOREM 1. *Let* $q_n(x)=\gamma'_n x^n+\cdots$ *and* $p_n(x)=\gamma_n x^n+\cdots$, *then*

$$q_n(x)=\frac{\gamma'_n}{\gamma_n}\,p_n(x)-\lambda q'_n(c)\,K^{(1,0)}_{n-1}(c, x) \tag{5}$$

holds with

$$\frac{\gamma'_n}{\gamma_n}=\sqrt{\frac{1+\lambda K^{(1,1)}_{n-1}(c, c)}{1+\lambda K^{(1,1)}_{n}(c, c)}}, \tag{6}$$

and

$$q'_n(c)=\frac{p'_n(c)}{\sqrt{(1+\lambda K^{(1,1)}_{n-1}(c, c))(1+\lambda K^{(1,1)}_{n}(c, c))}}. \tag{7}$$

We have used the abbreviations

$$K^{(1,0)}_n(x, y)=\sum_{k=0}^{n} p'_k(x)\,p_k(y)=\frac{\partial}{\partial x}\,K_n(x, y), \tag{8}$$

$$K^{(1,1)}_n(x, y)=\sum_{k=0}^{n} p'_k(x)\,p'_k(y)=\frac{\partial^2}{\partial x\,\partial y}\,K_n(x, y), \tag{9}$$

where

$$K_n(x, y)=\sum_{k=0}^{n} p_k(x)\,p_k(y)$$

is the well-known kernel associated with the orthonormal polynomials $p_n(x)$.

Proof. It is clear that a Fourier expansion

$$q_n(x)=\sum_{k=0}^{n} a_{k,n}\,p_k(x) \tag{10}$$

always exists and that the Fourier coefficients are given by

$$a_{k,n}=\int_{-1}^{1} q_n(x)\,p_k(x)\,d\mu(x)=\langle q_n, p_k\rangle-\lambda q'_n(c)\,p'_k(c).$$

If $k=n$ we can compate the leading coefficients to find that $a_{n,n}=\gamma_n'/\gamma_n$. When $k<n$ then by orthogonality we have $\langle q_n, p_k\rangle = 0$ so that $a_{k,n} = -\lambda q_n'(c)\, p_k'(c)$. This gives the expression (5). We now express the quantities γ_n' and $q_n'(c)$ in terms of the orthogonal polynomials $p_n(x)$. From (10) we find

$$\int_{-1}^{1} q_n^2(x)\, d\mu(x) = \sum_{k=0}^{n} a_{k,n}^2$$
$$= \left(\frac{\gamma_n'}{\gamma_n}\right)^2 + \lambda^2 [q_n'(c)]^2 \sum_{k=0}^{n-1} [p_k'(c)]^2. \tag{11}$$

On the other hand, we also have by the orthonormality of the $q_n(x)$ with respect to the Sobolev inner product

$$\int_{-1}^{1} q_n^2(x)\, d\mu(x) = 1 - \lambda [q_n'(c)]^2.$$

If we use this (11) and solve for $[q_n'(c)]^2$ then we find

$$[q_n'(c)]^2 = \frac{1-(\gamma_n'/\gamma_n)^2}{\lambda(1+\lambda K_{n-1}^{(1,1)}(c,c))}. \tag{12}$$

Another way to obtain an expression for $q_n'(c)$ is to take derivatives in (5) and evaluate at $x=c$, giving

$$q_n'(c) = \frac{\gamma_n'}{\gamma_n} p_n'(c) - \lambda q_n'(c)\, K_{n-1}^{(1,1)}(c,c).$$

Solving for $q_n'(c)$ gives

$$q_n'(c) = \frac{(\gamma_n'/\gamma_n)\, p_n'(c)}{1+\lambda K_{n-1}^{(1,1)}(c,c)}. \tag{13}$$

If we eliminate $q_n'(c)$ from Eqs. (12) and (13) then

$$\left(\frac{\gamma_n'}{\gamma_n}\right)^2 = \frac{1+\lambda K_{n-1}^{(1,1)}(c,c)}{1+\lambda K_n^{(1,1)}(c,c)}.$$

Insert this into either (12) or (13) to find (7). This completes the proof of the theorem. ▮

3. Relative Asymptotics for c off supp(μ)

We would like to obtain an asymptotic expression for the ratio $q_n(x)/p_n(x)$ because this will give information of how close the Sobolev orthogonal polynomials $q_n(x)$ are to the orthogonal polynomials $p_n(x)$. Recall that we have assumed that the measure μ is in Nevai's class $M(0, 1)$. This means that $\operatorname{supp}(\mu) = [-1, 1] \cup E$ with E a set which is at most denumerable and $E' \subset \{-1, 1\}$. As a consequence we have

$$\lim_{n \to \infty} \frac{p_{n-1}(x)}{p_n(x)} = \frac{1}{x + \sqrt{x^2 - 1}}, \tag{14}$$

$$\lim_{n \to \infty} \frac{1}{n} \frac{p_n'(x)}{p_n(x)} = \frac{1}{\sqrt{x^2 - 1}}, \tag{15}$$

$$\lim_{n \to \infty} \frac{p_{n-1}'(x)}{p_n'(x)} = \frac{1}{x + \sqrt{x^2 - 1}}, \tag{16}$$

uniformly for x on compact subsets of $\mathbb{C} \setminus \operatorname{supp}(\mu)$ [12, pp. 33–36]. The square root in the above formulas is such that $|x + \sqrt{x^2 - 1}| > 1$ whenever $x \in \mathbb{C} \setminus [-1, 1]$, which implies that $\sqrt{x^2 - 1} > 0$ for $x > 1$ and $\sqrt{x^2 - 1} < 0$ for $x < -1$. The following discrete version of l'Hôpital's rule turns out to be very convenient (see, e.g., [5, Sect. 147, p. 414]).

LEMMA 1 (Stolz Criterion). *Let x_n and y_n $(n = 0, 1, 2, ...)$ be real sequences and suppose that y_n $(n = 0, 1, 2, ...)$ is monotone and that $y_n \neq 0$ for all n. If*

$$\lim_{n \to \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L \in \mathbb{R} \cup \{\pm \infty\}$$

exists, then

$$\lim_{n \to \infty} \frac{x_n}{y_n} = L$$

provided either

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0,$$

or

$$\lim_{n \to \infty} y_n = \pm \infty.$$

We are now ready to proof the main theorem in this section:

THEOREM 2. *Let $q_n(x)$ be the orthonormal polynomials for the inner* *product* (3) *and* $p_n(x)$ *the orthonormal polynomials for the measure* μ. *If* $\mu \in M(0, 1)$ *and* $c \in \mathbb{R} \backslash \operatorname{supp}(\mu)$ *then*

$$\lim_{n \to \infty} \frac{q_n(x)}{p_n(x)} = \frac{1}{|c + \sqrt{c^2 - 1}|} \times \left(1 - \frac{\sqrt{c^2 - 1}}{x + \sqrt{x^2 - 1}} \frac{(x + \sqrt{x^2 - 1}) - (c + \sqrt{c^2 - 1})}{x - c}\right), \tag{17}$$

uniformly for x *on compact subsets of* $\mathbb{C} \backslash (\operatorname{supp}(\mu) \cup \{c\})$.

Proof. It is clear that $K_n^{(1,1)}(c, c)$ is an increasing sequence and by (16) this sequence tends to infinity. By the Stolz criterion and (16) we thus find

$$\lim_{n \to \infty} \frac{1 + \lambda K_{n-1}^{(1,1)}(c, c)}{1 + \lambda K_n^{(1,1)}(c, c)} = \frac{1}{(c + \sqrt{c^2 - 1}))^2}. \tag{18}$$

The identity

$$\frac{\lambda [p_n'(c)]^2}{1 + \lambda K_n^{(1,1)}(c, c)} = 1 - \frac{1 + \lambda K_{n-1}^{(1,1)}(c, c)}{1 + \lambda K_n^{(1,1)}(c, c)}$$

implies that

$$\lim_{n \to \infty} \frac{[p_n'(c)]^2}{1 + \lambda K_n^{(1,1)}(c, c)} = \frac{1}{\lambda}\left(1 - \frac{1}{(c + \sqrt{c^2 - 1})^2}\right) = \frac{2}{\lambda} \frac{\sqrt{c^2 - 1}}{c + \sqrt{c^2 - 1}}. \tag{19}$$

Next we will obtain a suitable expression for $K_{n-1}^{(0,1)}(x, c)$. From the Christoffel–Darboux formula

$$K_{n-1}(x, y) = \sum_{k=0}^{n-1} p_k(x)\, p_k(y) = a_n^0 \frac{p_n(x)\, p_{n-1}(y) - p_n(y)\, p_{n-1}(x)}{x - y}$$

we find by taking a derivative with respect to y that

$$K_{n-1}^{(0,1)}(x, c) = a_n^0 \frac{p_n(x)\, p_{n-1}'(c) - p_n'(c)\, p_{n-1}(x)}{x - c} + a_n^0 \frac{p_n(x)\, p_{n-1}(c) - p_n(c)\, p_{n-1}(x)}{(x - c)^2}, \tag{20}$$

and if we use (14)–(16) then this gives

$$\lim_{n \to \infty} \frac{K_{n-1}^{(0,1)}(x, c)}{p_n(x)\, p_{n-1}'(c)} = \frac{1}{2} \frac{1}{x + \sqrt{x^2 - 1}} \frac{(x + \sqrt{x^2 - 1}) - (c + \sqrt{c^2 - 1})}{x - c}. \tag{21}$$

The asymptotic result (17) now follows by dividing both sides of (5) by $p_n(x)$ (this can always be done because all the zeros of $p_n(x)$ accumulate at $\text{supp}(\mu)$) and by letting n tend to infinity, using (18)–(21). ∎

The asymptotic formula (17) should be compared with [12, p. 132, Lemma 16]: both limits can be shown to be identical. Note that the limit in (17) is independent of λ, which shows that the asymptotic behaviour in (17) is not uniform in $\lambda > 0$ since for $\lambda = 0$ the ratio $q_n(x)/p_n(x)$ is always 1. By going through the proof with $x = c$ (now use the confluent form for $K_{n-1}^{(1,0)}(c, c)$) we have

$$\lim_{n\to\infty} \frac{q_n(c)}{p_n(c)} = 0.$$

From (7) one can also find

$$\lim_{n\to\infty} q_n'(c)\, p_n'(c) = \frac{2}{\lambda}\, |\sqrt{c^2-1}|, \tag{22}$$

and now the parameter λ is present. This formula should be compared to the corresponding expression in the case where a mass point distribution is added to μ [12, last formula on p. 132].

4. Relative Asymptotics for c on $\text{supp}(\mu)$

The asymptotic behaviour in the previous section was possible because when $c \notin \text{supp}(\mu)$ then one has for $\mu \in M(0, 1)$

$$\lim_{n\to\infty} \frac{[p_n'(c)]^2}{K_{n-1}^{(1,1)}(c, c)} = 2\sqrt{c^2-1}\,(c+\sqrt{c^2-1}).$$

A similar result for $c \in \text{supp}(\mu)$ needs more work, but recall that for $\mu \in M(0, 1)$ and $x \in \text{supp}(\mu)$ we have

$$\lim_{n\to\infty} \frac{p_n^2(x)}{K_{n-1}(x, x)} = 0, \tag{23}$$

[12, pp. 31–32, Theorem 11; 13]. We first prove a similar result but for the derivatives of the orthogonal polynomials. In order to prove this result we introduce the measure μ_2 for which $d\mu_2(x) = (x-c)^2\, d\mu(x)$, so that μ_2 is absolutely continuous with respect to μ. Let us first mention some relevant facts about the orthonormal polynomials associated with this measure.

LEMMA 2. *Let $p_n(x;\mu_2)$ be the orthonormal polynomials with respect to μ_2 and denote the related kernel by $K_n(x, y; \mu_2)$. If $p_n(x)$ and $K_n(x, y)$ are the orthonormal polynomials and the kernel for the measure μ then*

$$(x-c)\,p_{n-1}(x;\mu_2)=\sqrt{\frac{K_{n-1}(c,c)}{K_n(c,c)}}\left(p_n(x)-\frac{p_n(c)}{K_{n-1}(c,c)}K_{n-1}(x,c)\right), \qquad (24)$$

and

$$(x-c)(y-c)\,K_{n-1}(x,y;\mu_2)=K_n(x,y)-\frac{K_n(x,c)\,K_n(y,c)}{K_n(c,c)}. \qquad (25)$$

Proof. Denote by $\mathbf{P}_n$ the linear space of polynomials of degree at most n, then the system $\{(x-c)\,p_k(x;\mu_2):k=0,1,...,n-1\}\cup\{K_n(x,c)\}$ is an orthogonal basis in $L_2(\mu)$ for $\mathbf{P}_n$. This means that in $L_2(\mu)$ we have

$$\mathbf{P}_n=(x-c)\,\mathbf{P}_{n-1}\oplus_\perp L\{K_n(x,c)\} \qquad (26)$$

$$=(x-c)\,\mathbf{P}_{n-2}\oplus_\perp L\{K_n(x,c),(x-c)\,p_{n-1}(x;\mu_2)\}, \qquad (27)$$

where $L\{f_1,...,f_k\}$ is the linear space spanned by the functions $f_1,...,f_k$. Of course, $\{p_k(x):k=0,1,...,n\}$ is also an orthogonal basis in $L_2(\mu)$ for $\mathbf{P}_n$ so that

$$\mathbf{P}_n=\mathbf{P}_{n-1}\oplus_\perp L\{p_n(x)\}$$

holds in $L_2(\mu)$, and by using (26) this gives

$$\mathbf{P}_n=(x-c)\,\mathbf{P}_{n-2}\oplus_\perp L\{p_n(x),K_{n-1}(x,c)\}.$$

This combined with (27) means that

$$L\{K_n(x,c),(x-c)\,p_{n-1}(x;\mu_2)\}=L\{p_n(x),K_{n-1}(x,c)\},$$

and in particular

$$(x-c)\,p_{n-1}(x;\mu_2)=Ap_n(x)+BK_{n-1}(x,c).$$

Setting $x=c$ gives $B=-Ap_n(c)/K_{n-1}(c,c)$. The coefficient A can be found by computing the $L_2(\mu)$ norm. This gives (24). In order to prove (25) we observe that—for y fixed—$(x-c)(y-c)\,K_{n-1}(x,y;\mu_2)-K_n(x,y)\in\mathbf{P}_n$, and thus by (26) we have

$$(x-c)(y-c)\,K_{n-1}(x,y;\mu_2)-K_n(x,y)$$
$$=\sum_{k=0}^{n-1}c_k(x-c)\,p_k(x;\mu_2)+c_nK_n(x,c),$$

where for $k < n$

$$c_k = \int (x-c)^2 (y-c) K_{n-1}(x, y; \mu_2) p_k(x; \mu_2) d\mu(x)$$

$$- \int K_n(x, y)(x-c) p_k(x; \mu_2) d\mu(x).$$

By using the reproducing property of the kernels we find

$$c_k = (y-c) p_k(y; \mu_2) - (y-c) p_k(y; \mu_2) = 0, \qquad k < n.$$

The coefficient c_n can easily be obtained by putting $x = c$, giving (25). ∎

Very often one uses a result by Christoffel to express $(x-c)^2 p_n(x; \mu_2)$ as a Fourier series in $p_k(x)$ with a finite number of terms [15, pp. 29–31, Sect. 2.5]. We have chosen an approach to express $(x-c) p_n(x; \mu_2)$ as a Fourier series in $p_k(x)$ but with $n+1$ terms. This approach—on Jordan arcs—can already be found in [6].

Now we can prove the following result about the growth of the derivatives of orthogonal polynomials relative to their sums, which shows that the important results on the relative growth of orthogonal polynomials in $M(0, 1)$ obtained by Nevai *et al.* [12, 13] also hold for the derivatives of the orthogonal polynomials.

THEOREM 3. *Suppose* $\mu \in M(0, 1)$ *and* $c \in [-1, 1]$ *then*

$$\lim_{n \to \infty} \frac{[p_n'(c)]^2}{K_{n-1}^{(1,1)}(c, c)} = 0. \tag{28}$$

Proof. Since $\mu \in M(0, 1)$ it follows that also $\mu_2 \in M(0, 1)$ [12, p. 68, Theorem 20]. From (23) we have

$$\lim_{n \to \infty} \frac{p_{n-1}^2(c; \mu_2)}{K_{n-2}(c, c; \mu_2)} = 0. \tag{29}$$

By taking derivatives in (24) we obtain

$$p_{n-1}(c; \mu_2) = \sqrt{\frac{K_{n-1}(c, c)}{K_n(c, c)}} \left(p_n'(c) - \frac{p_n(c)}{K_{n-1}(c, c)} K_{n-1}^{(1,0)}(c, c) \right)$$

and similarly from (25)

$$K_{n-2}(c, c; \mu_2) = K_{n-1}^{(1,1)}(c, c) - \frac{[K_{n-1}^{(1,0)}(c, c)]^2}{K_{n-1}(c, c)} \leqslant K_{n-1}^{(1,1)}(c, c). \tag{30}$$

We thus have

$$\frac{p_{n-1}^2(c;\mu_2)}{K_{n-2}(c,c;\mu_2)} \geqslant c_n \frac{(p_n'(c)-(p_n(c)/K_{n-1}(c,c))\,K_{n-1}^{(1,0)}(c,c))^2}{K_{n-1}^{(1,1)}(c,c)}, \tag{31}$$

where

$$c_n = \frac{1}{1+p_n^2(c)/K_{n-1}(c,c)}.$$

From (23) it follows that $c_n \to 1$ as $n \to \infty$. By the Cauchy–Schwarz inequality

$$|K_{n-1}^{(1,0)}(c,c)|^2 \leqslant K_{n-1}^{(1,1)}(c,c)\, K_{n-1}(c,c),$$

so that

$$\frac{|p_n(c)|}{K_{n-1}(c,c)} \frac{|K_{n-1}^{(1,0)}(c,c)|}{\sqrt{K_{n-1}^{(1,1)}(c,c)}} \leqslant \frac{|p_n(c)|}{\sqrt{K_{n-1}(c,c)}} \to 0, \tag{32}$$

where we used (23) for $n \to \infty$. The result now follows by combining (29)–(32). ∎

We now have all the results needed for proving the relative asymptotic behaviour of the Sobolev orthogonal polynomials, for $c \in [-1, 1]$:

THEOREM 4. *Let $q_n(x) = \gamma_n' x^n + \cdots$ be the orthonormal polynomials for the inner product (3) and $p_n(x) = \gamma_n x^n + \cdots$ the orthonormal polynomials for the measure μ. Suppose $\mu \in M(0, 1)$ and $c \in [-1, 1]$, then*

$$\lim_{n\to\infty} \frac{\gamma_n'}{\gamma_n} = 1, \tag{33}$$

and

$$\lim_{n\to\infty} q_n'(c)\, p_n'(c) = 0. \tag{34}$$

Moreover we have the relative asymptotic behaviour

$$\lim_{n\to\infty} \frac{q_n(x)}{p_n(x)} = 1 \tag{35}$$

uniformly for x on compact subsets of $\mathbb{C}\setminus\operatorname{supp}(\mu)$.

Proof. The result in (33) follows immediately from (6) and (28). In a similar way (34) follows from (7) and (28). Finally

$$\frac{q_n(x)}{p_n(x)}=\frac{\gamma_n'}{\gamma_n}-\lambda q_n'(c)\,\frac{K_{n-1}^{(1,0)}(c,x)}{p_n(x)}.$$

By Cauchy–Schwarz we have

$$|K_{n-1}^{(1,0)}(c,x)|^2\leqslant K_{n-1}(x,x)\,K_{n-1}^{(1,1)}(c,c)$$

and since $K_{n-1}(x,x)/p_n^2(x)$ is bounded on compact subsets of $\mathbb{C}\backslash\text{supp}(\mu)$, the result follows from (28). ∎

We have only been considering asymptotic formulas for $x\notin\text{supp}(\mu)$. When $c\in[-1,1]$ then we also find an asymptotic formula on the oscillatory region:

THEOREM 5. *Suppose that* $c\in[-1,1]$. *If* $\mu\in M(0,1)$ *and if there exists a function* $\psi(x)$ *such that* $\psi(x)\,p_n(x)$ *is uniformly bounded on* $\text{supp}(\mu)$, *then*

$$\lim_{n\to\infty}\psi(x)[q_n(x)-p_n(x)]=0, \tag{36}$$

uniformly on closed sets of $\text{supp}(\mu)\backslash\{c\}$.

Proof. Clearly

$$\psi(x)[q_n(x)-p_n(x)]=\psi(x)\left[q_n(x)-\frac{\gamma_n'}{\gamma_n}\,p_n(x)\right]+\psi(x)\,p_n(x)\left[\frac{\gamma_n'}{\gamma_n}-1\right],$$

so that by (33) it is sufficient to show that

$$\lim_{n\to\infty}\psi(x)\left[q_n(x)-\frac{\gamma_n'}{\gamma_n}\,p_n(x)\right]=0,$$

uniformly on closed sets of $\text{supp}(\mu)\backslash\{c\}$, which by (5) is equivalent to showing that

$$\lim_{n\to\infty}q_n'(c)\,\psi(x)\,K_{n-1}^{(1,0)}(c,x)=0, \tag{37}$$

uniformly on closed sets of $\text{supp}(\mu)\backslash\{c\}$. If we use (20) then (37) will follow if we can show that

$$\lim_{n\to\infty} q_n'(c)\, p_n'(c) = 0,$$

$$\lim_{n\to\infty} q_n'(c)\, p_{n-1}'(c) = 0,$$

$$\lim_{n\to\infty} q_n'(c)\, p_n(c) = 0,$$

$$\lim_{n\to\infty} q_n'(c)\, p_{n-1}(c) = 0.$$

The first of these relations is already given by (34) and the second asymptotic formula follows in a similar manner. We will only show the third of these asymptotic formulas (the fourth can be shown using the same reasoning). By using expression (7) we have

$$q_n'(c)\, p_n(c) = \frac{p_n'(c)\, p_n(c)}{\sqrt{1+\lambda K_{n-1}^{(1,1)}(c,c)}\,\sqrt{1+\lambda K_n^{(1,1)}(c,c)}},$$

and by Theorem 3 we need only consider the ratio

$$\frac{p_n(c)}{\sqrt{1+\lambda K_n^{(1,1)}(c,c)}} = \frac{p_n(c)}{\sqrt{1+\lambda K_{n-1}(c,c)}}\,\frac{\sqrt{1+\lambda K_{n-1}(c,c)}}{\sqrt{1+\lambda K_n^{(1,1)}(c,c)}}.$$

From (30) we obtain

$$\frac{K_{n-1}(c,c)}{K_n^{(1,1)}(c,c)} \leqslant \frac{K_{n-1}(c,c)}{K_{n-1}(c,c;\mu_2)},$$

where $d\mu_2(x) = (x-c)^2\, d\mu(x)$. By a result of Nevai [12, p. 78, Theorem 6] we have

$$\lim_{n\to\infty} \frac{K_{n-1}(x,x)}{K_{n-1}(x,x;\mu_2)} = (x-c)^2$$

for every $x \in \operatorname{supp}(\mu)$ and in particular the limit is zero for $x=c$. This means that

$$\lim_{n\to\infty} \frac{1+\lambda K_{n-1}(c,c)}{1+\lambda K_n^{(1,1)}(c,c)} = 0,$$

and from this we obtain

$$\lim_{n\to\infty} \frac{p_n(c)}{\sqrt{1+\lambda K_n^{(1,1)}(c,c)}} = 0,$$

from which the theorem follows. ∎

These asymptotic formulas should again be compared to those corresponding to adding a mass point distribution at c to the measure μ [12, p. 132].

The asymptotic behaviour of $p_n(x)$ when x is near $c \in [-1, 1]$ remains an open problem. Also the behaviour of $p_n(x)$ on the support of μ when $c \notin \operatorname{supp}(\mu)$ remains open. The asymptotics when $c \in \operatorname{supp}(\mu) \setminus [-1, 1]$ is more delicate and will be considered elsewhere.

5. Recurrence Relation

The important property of the Sobolev inner product

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, d\mu(x) + \sum_{k=1}^{m} \lambda_k f^{(k)}(c)\, g^{(k)}(c) \tag{38}$$

is the fact that

$$\langle (x-c)^{m+1} f, g \rangle = \langle f, (x-c)^{m+1} g \rangle \tag{39}$$

which expresses self-adjointness of multiplication by $(x-c)^{m+1}$. This property is easily verified. From it we obtain the following result [11]:

Theorem 6 (Marcellán and Ronveaux). *The orthonormal polynomials $q_n(x)$ with Sobolev inner product* (38) *satisfy a* $(2m+3)$*-term recurrence relation*

$$(x-c)^{m+1} q_n(x) = \sum_{j=n-m-1}^{n+m+1} c_{j,n} q_j(x), \tag{40}$$

where

$$c_{j,n} = \langle (x-c)^{m+1} q_n, q_j \rangle = \langle q_n, (x-c)^{m+1} q_j \rangle = c_{n,j}. \tag{41}$$

Consider the infinite matrix

$$C = (c_{i,j})_{i,j \geq 0},$$

then C is a banded and symmetric matrix with band width $2m+3$. For the case when $m = 1$ we have a five-term recurrence relation

$$(x-c)^2 q_n(x) = a_{n+2} q_{n+2}(x) + b_{n+1} q_{n+1}(x) + c_n q_n(x) + b_n q_{n-1}(x) + a_n q_{n-2}(x), \tag{42}$$

with initial values

$$q_{-2}(x)=0, \qquad q_{-1}(x)=0, \qquad q_0(x)=p_0(x),$$
$$q_1(x)=\sqrt{\frac{1}{1+\lambda[p_1'(c)]^2}}\, p_1(x).$$

The recurrence coefficients are

$$a_n=\langle (x-c)^2 q_n, q_{n-2}\rangle, \qquad n\geqslant 2,$$
$$b_n=\langle (x-c)^2 q_n, q_{n-1}\rangle, \qquad n\geqslant 1,$$
$$c_n=\langle (x-c)^2 q_n, q_n\rangle, \qquad n\geqslant 0.$$

We show that under appropriate conditions the pentadiagonal matrix C for this recurrence relation is a compact perturbation of the pentadiagonal matrix $(J-c)^2$, where J is the tridiagonal (Jacobi) matrix associated with the orthogonal polynomials with measure μ,

$$J=\left(\int xp_n(x)\, p_m(x)\, d\mu(x)\right)_{n,m\geqslant 0}.$$

THEOREM 7. *The recurrence coefficients in* (42) *of orthonormal polynomials with respect to the inner product* (3) *are given by*

$$a_n=\frac{a_{n-2,n-2}}{a_{n,n}}\,((J-c)^2)_{n-2,n}, \tag{43}$$

$$b_n=\frac{a_{n-1,n-1}}{a_{n,n}}\,((J-c)^2)_{n-1,n}+\frac{a_{n-2,n-1}}{a_{n,n}}\,((J-c)^2)_{n-2,n}$$
$$-\frac{a_{n,n+1}}{a_{n,n}}\frac{a_{n-1,n-1}}{a_{n+1,n+1}}\,((J-c)^2)_{n-1,n+1}, \tag{44}$$

$$c_n=((J-c)^2)_{n,n}+\frac{a_{n-1,n}}{a_{n,n}}\,((J-c)^2)_{n-1,n}-\frac{a_{n,n+1}}{a_{n+1,n+1}}\,((J-c)^2)_{n,n+1}$$
$$+\frac{a_{n-2,n}}{a_{n,n}}\,((J-c)^2)_{n,n-2}$$
$$+\left(\frac{a_{n+1,n+2}}{a_{n+2,n+2}}\frac{a_{n,n+1}}{a_{n+1,n+1}}-\frac{a_{n,n+2}}{a_{n+2,n+2}}\right)((J-c)^2)_{n+2,n}$$
$$-\frac{a_{n-1,n}}{a_{n,n}}\frac{a_{n,n+1}}{a_{n+1,n+1}}\,((J-c)^2)_{n+1,n-1}, \tag{45}$$

where $a_{n,n} = \gamma'_n/\gamma_n$ *and* $a_{k,n} = -\lambda q'_n(c)\, p'_k(c)$ $(k<n)$. *If* $\mathrm{supp}(\mu) = [-1, 1]$ *and* $\mu \in M(0, 1)$ *then for every* $c \in \mathbb{R}$

$$\lim_{n\to\infty} a_n = \tfrac{1}{4}, \qquad \lim_{n\to\infty} b_n = -c, \qquad \lim_{n\to\infty} c_n = \frac{1+2c^2}{2}. \tag{46}$$

Proof. The orthonormal polynomials with respect to the measure μ satisfy the three term recurrence relation

$$xp_n(x) = a^0_{n+1}\, p_{n+1}(x) + b^0_n\, p_n(x) + a^0_n\, p_{n-1}(x)$$

which is equivalent to

$$Jp = xp,$$

where J is the Jacobi matrix and $p = (p_0, p_1, p_2, \ldots)'$ is the column vector containing the orthonormal polynomials. From this we find

$$(J-c)^2\, p = (x-c)^2\, p,$$

which in turn is equivalent to

$$\begin{aligned}(x-c)^2\, p_n(x) = {} & ((J-c)^2)_{n,n+2}\, p_{n+2}(x) + ((J-c)^2)_{n,n+1}\, p_{n+1}(x) \\ & + ((J-c)^2)_{n,n}\, p_n(x) \\ & + ((J-c)^2)_{n,n-1}\, p_{n-1}(x) + ((J-c)^2)_{n,n-2}\, p_{n-2}(x). \end{aligned} \tag{47}$$

The recurrence relation (40), with $m = 1$, is equivalent to

$$Cq = (x-c)^2\, q,$$

where C is the infinite matrix with entries given by (41) and $q = (q_0, q_1, q_2, \ldots)'$. The Fourier expansion (10) can be written as $q = Ap$, with A the lower triangular matrix containing the $a_{i,j}$. If we multiply the equation $(x-c)\,p = (J-c)\,p$ on the left by A, then

$$(x-c)q = A(J-c)\,A^{-1}q,$$

from which one easily obtains

$$(x-c)^2\, q = A(J-c)^2\, A^{-1}q,$$

so that $C = A(J-c)^2\, A^{-1}$. We know that C is a symmetric pentadiagonal matrix, so we can compute the entries $a_n = C_{n-2,n}$, $b_n = C_{n-1,n}$, and $c_n = C_{n,n}$ by using the formulas

$$(A^{-1})_{n,n} = \frac{1}{a_{n,n}}, \qquad (A^{-1})_{n+1,n} = -\frac{a_{n,n+1}}{a_{n,n}a_{n+1,n+1}},$$

$$(A^{-1})_{n+2,n} = -\frac{a_{n,n+2}}{a_{n,n}a_{n+2,n+2}} + \frac{a_{n+1,n+2}a_{n,n+1}}{a_{n,n}a_{n+1,n+1}a_{n+2,n+2}},$$

giving (45).

The asymptotic behaviour given by (46) follows because when $\mu \in M(0, 1)$ then

$$\lim_{n \to \infty} ((J-c)^2)_{n,n+2} = \tfrac{1}{4},$$

$$\lim_{n \to \infty} ((J-c)^2)_{n,n+1} = -c,$$

$$\lim_{n \to \infty} ((J-c)^2)_{n,n} = \frac{1+2c^2}{2}.$$

The behaviour of the factors $a_{k,n}$ follows from (6), (16), (18), and (22) when $c \notin \text{supp}(\mu)$; when $c \in [-1, 1]$ then one also uses (7), (28), and (33). ∎

The restriction that $\text{supp}(\mu) = [-1, 1]$ can be removed but needs a more delicate analysis when $c \in \text{supp}(\mu) \setminus [-1, 1]$.

Acknowledgments

We thank Manolo Alfaro and Marisa Rezola for some fruitful discussions and useful suggestions. We are also very grateful to Alphonse Magnus and the referees whose suggestions resulted in a more elegant proof of Theorem 7.

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